Gravitational Acceleration and the Curvature Distortion of Spacetime

William C. Daywitt
National Institute for Standards and Technology (retired), Boulder, Colorado
E-mail: wcdaywitt@me.com

The Crothers solution to the Einstein vacuum field consists of a denumerable infinity of Schwarzschild-like metrics that are non-singular everywhere except at the point mass itself. When the point-mass distortion from the Planck vacuum (PV) theory is inserted into the Crothers calculations, the combination yields a composite model that is physically transparent. The resulting static gravitational field using the Crothers metrics is calculated and compared to the Newtonian gravitational field and the gravitational field associated with the black hole model.

1 Newtonian Introduction

When a test mass m' travels in the gravitational field of a point mass m situated at $r = 0$, the Newtonian theory of gravity predicts that the acceleration experienced by the test mass

$$\frac{d^2r}{dt^2} = -\frac{mG}{r^2} \tag{1}$$

is *independent of the mass* m'. In this theory the relative magnitudes of m' and m are arbitrary and lead to the following equation for the magnitude of the gravitational force between the two masses

$$\frac{m'mG}{r^2} = \frac{(m'c^2/r)(mc^2/r)}{c^4/G} = \left(\frac{m'c^2/r}{c^4/G}\right)\left(\frac{mc^2/r}{c^4/G}\right)\frac{c^4}{G} \tag{2}$$

when expressed in terms of the ratio c^4/G.

In the PV theory [1] the force mc^2/r represents the curvature distortion the mass m exerts on the PV state (and hence on spacetime), and the ratio

$$\frac{c^4}{G} = \frac{m_* c^2}{r_*} \tag{3}$$

represents the maximum such curvature force, where m_* and r_* are the mass and Compton radius of the Planck particles constituting the PV. The corresponding relative curvature force is represented by the n-ratio

$$n_r \equiv \frac{mc^2/r}{c^4/G} = \frac{mc^2/r}{m_* c^2/r_*} \tag{4}$$

which is a direct measure of the curvature distortion exerted on spacetime and the PV by the point mass. Since the minimum distortion is 0 ($m = 0$ or $r \to \infty$) and the maximum is 1, the n-ratio is physically restricted to the range $0 \leq n_r \leq 1$ as are the equations of general relativity [2].

The important fiducial point at $n_{r_s} = 0.5$ is the Schwarzschild radius $r_s = 2mr_*/m_*$, where

$$rn_r = \frac{mc^2}{m_* c^2/r_*} = r_s n_{r_s} = 0.5 r_s\,. \tag{5}$$

The acceleration (1) can now be expressed exclusively in terms of the relative curvature distortion n_r:

$$a(n_r) = -\frac{d^2r}{dt^2} = \frac{mc^4}{r^2c^4/G} = \frac{c^2}{r}\frac{mc^2/r}{m_* c^2/r_*} = \frac{c^2}{r}n_r = \frac{c^2}{rn_r}n_r^2 = \frac{2c^2}{r_s}n_r^2 \tag{6}$$

whose normalized graph $a/(2c^2/r_s)$ is plotted in the first figure.

2 Affine Connection

The conundrum posed by equation (1), that the acceleration of the test particle is independent of its mass m', is the principle motivation behind the general theory of relativity [3, p. 4]; an important ramification of which is that, in a free-falling local reference frame, the acceleration vanishes as in equation (7). That result leads to the following development. Given the two coordinate systems $x^\mu = x^\mu(\xi^\nu)$ and $\xi^\mu = \xi^\mu(x^\nu)$ and the differential equation

$$\frac{d^2\xi^\mu}{d\tau^2} = 0 \tag{7}$$

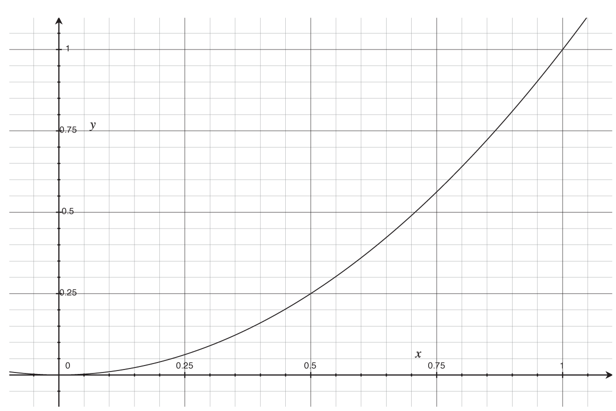

Fig. 1: The graph plots the normalized Newtonian acceleration $a/(2c^2/r_s)$ as a function of n_r ($0 \leq n_r \leq 1$).

applying the chain law to the differentials gives

$$\frac{d^2\xi^\mu}{d\tau^2} = \frac{\partial\xi^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2\xi^\mu}{\partial x^\alpha \partial x^\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} = 0\,. \tag{8}$$

Then using

$$x^\alpha(\xi^\mu(x^\beta)) = x^\alpha \qquad \Longrightarrow \qquad \frac{\partial x^\beta}{\partial \xi^\mu}\frac{\partial \xi^\mu}{\partial x^\nu} = \delta^\beta_\nu \tag{9}$$

to eliminate the coefficient of $d^2x^\nu/d\tau^2$ in (8) leads to

$$\frac{d^2x^\beta}{d\tau^2} + \frac{\partial x^\beta}{\partial\xi^\mu}\frac{\partial^2\xi^\mu}{\partial x^\alpha\partial x^\nu}\frac{dx^\alpha}{d\tau}\frac{dx^\nu}{d\tau} = 0\,. \tag{10}$$

Rearranging indices in (10) finally yields

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho}u^\nu u^\rho = 0 \tag{11}$$

where $u^\mu = dx^\mu/d\tau$ is a typical component of the test-mass 4-velocity and

$$\Gamma^\mu_{\nu\rho} \equiv \frac{\partial x^\mu}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\nu\partial x^\rho} \tag{12}$$

is the *affine connection*. The affine connection vanishes when there is no gravitational distortion; so for the point mass m, it should be solely a function of the curvature distortion n_r given by (4).

The affine connection can be related to the the metric coefficients $g_{\alpha\beta}$ via [3, p. 7]

$$\Gamma^\mu_{\nu\rho} = \frac{g^{\mu\alpha}}{2}\left[\frac{\partial g_{\rho\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\rho} - \frac{\partial g_{\nu\rho}}{\partial x^\alpha}\right] \tag{13}$$

which, for a metric with no cross terms ($g^{\alpha\beta} = 0$ for $\alpha \neq \beta$), reduces to

$$\frac{2\Gamma^1_{\nu\rho}}{g^{11}} = \frac{\partial g_{\rho 1}}{\partial x^\nu} + \frac{\partial g_{\nu 1}}{\partial x^\rho} - \frac{\partial g_{\nu\rho}}{\partial x^1} \tag{14}$$

with $\mu = 1$ for example.

Since only radial effects are of interest in the present paper, only the x^0 and x^1 components of the spherical polar coordinate system $(x^\mu) = (x^0, x^1, x^2, x^3) = (ct, r, \theta, \phi)$ are required. Then the affine connection in (11) for the $\mu = 1$ component reduces to

$$\frac{du^1}{d\tau} = -\Gamma^1_{\nu\rho}u^\nu u^\rho$$
$$= -\left[\Gamma^1_{00}(u^0)^2 + 2\Gamma^1_{01}u^0u^1 + \Gamma^1_{11}(u^1)^2\right] \tag{15}$$

which under static conditions ($u^1 = dr/d\tau = 0$ for the test mass) produces

$$\frac{du^1}{d\tau} = -\Gamma^1_{00}(u^0)^2\,. \tag{16}$$

In the spherical system with $d\theta = d\phi = 0$, the metric becomes

$$ds^2 = c^2d\tau^2 = g_{00}\,c^2dt^2 + g_{11}\,dr^2 \tag{17}$$

where g_{00} and g_{11} are functions of the coordinate radius $x^1 = r$. Under these conditions the only non-zero affine connections from (14) are:

$$\Gamma^0_{10} = \Gamma^0_{01} = \frac{g^{00}}{2}\frac{\partial g_{00}}{\partial x^1} \tag{18}$$

$$\Gamma^1_{00} = \frac{-g^{11}}{2}\frac{\partial g_{00}}{\partial x^1} \qquad \text{and} \qquad \Gamma^1_{11} = \frac{g^{11}}{2}\frac{\partial g_{11}}{\partial x^1}\,. \tag{19}$$

Using (17), the velocity u^0 can be calculated from

$$cd\tau = g_{00}^{1/2}dx^0\left[1 + \left(\frac{g_{11}}{g_{00}}\right)\left(\frac{dr/dt}{c}\right)^2\right]^{1/2} \tag{20}$$

which for static conditions ($dr/dt = 0$) leads to

$$u^0 = \frac{dx^0}{d\tau} = \frac{c}{g_{00}^{1/2}}\,. \tag{21}$$

Inserting (21) into (16) gives

$$\frac{du^1}{d\tau} = -\frac{c^2\Gamma^1_{00}}{g_{00}} = \frac{c^2}{g_{00}}\left(\frac{g^{11}}{2}\frac{\partial g_{00}}{\partial r}\right) \tag{22}$$

along with its covariant twin

$$\frac{du_1}{d\tau} = g_{11}\frac{du^1}{d\tau}$$
$$= \frac{g_{11}c^2}{g_{00}}\left(\frac{g^{11}}{2}\frac{\partial g_{00}}{\partial r}\right) = \frac{c^2}{g_{00}}\left(\frac{\partial g_{00}}{2\partial r}\right). \tag{23}$$

Then combining (22) and (23) leads to the static acceleration

$$\left|\frac{du^1}{d\tau}\frac{du_1}{d\tau}\right|^{1/2} = \left(-g^{11}\right)^{1/2}\left(\frac{c^2}{g_{00}}\right)\left(\frac{\partial g_{00}}{2\partial r}\right). \tag{24}$$

3 Static Acceleration

The metric coefficients g_{00} and g^{11} for a point mass m at $r = 0$ are given by (A6) and (A7) in the Appendix. After some straightforward manipulations, (24) leads to the (normalized) static gravitational acceleration ($0 \leq n_r \leq 1$)

$$\frac{a_n(n_r)}{2c^2/r_s} = \left|\frac{(du^1/d\tau)(du_1/d\tau)}{(2c^2/r_s)^2}\right|^{1/2}$$
$$= \frac{n_r^2}{(1 - r_s/R_n)^{1/2}(1 + 2^n n_r^n)^{2/n}} \tag{25}$$
$$= \frac{n_r^2}{[(1 + 2^n n_r^n)^{1/n} - 2n_r]^{1/2}(1 + 2^n n_r^n)^{3/2n}} \tag{26}$$
$$= \frac{n_r^2}{[(1 + 1/2^n n_r^n)^{1/n} - 1]^{1/2}(2n_r)^{1/2}(1 + 2^n n_r^n)^{3/2n}} \tag{27}$$

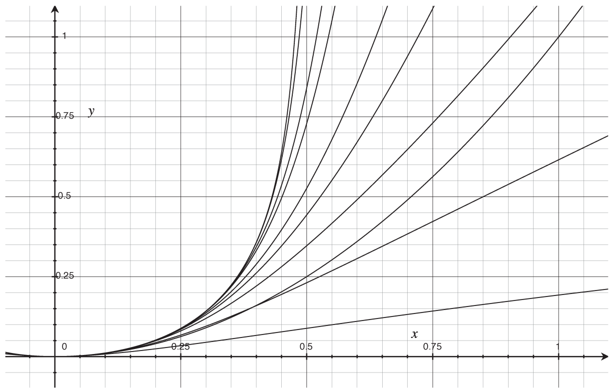

Fig. 2: The graph plots $a_n/(2c^2/r_s)$ as a function of n_r for the indices $n = 1, 2, 3, 4, 5, 8, 10, 20, 40$ from bottom-to-top of the graph. The curve that intersects (1,1) is the normalized Newtonian acceleration from (6). The $n = 3$ curve is the original Schwarzschild result [5] $(0 \leq n_r \leq 1)$.

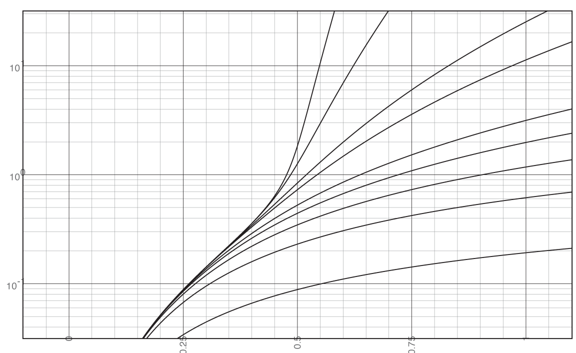

Fig. 3: The graph is a lin-log plot of $a_n/(2c^2/r_s)$ as a function of n_r for the indices $n = 1, 2, 3, 4, 5, 8, 10, 20, 40$ from bottom-to-top of the graph $(0 \leq n_r \leq 1)$.

in terms of the relative curvature force n_r, all of which vanish for $n_r = 0$. Formally, the acceleration in the denominator on the left of (25)

$$\frac{\Delta v}{\Delta t} = \frac{c}{(r_s - r_s/2)/c} = \frac{2c^2}{r_s} \tag{28}$$

is the acceleration of a test mass starting from rest at $r = r_s$ $(n_r = 0.5)$ and accelerating to the speed of light c in its fall to $r_s/2$ $(n_r = 1)$ in the time interval $(r_s - r_s/2)/c$.

The limits of (26) and (27) as $n \to \infty$ are easily seen to be

$$\frac{a_\infty(n_r)}{2c^2/r_s} = \begin{cases} n_r^2/(1-2n_r)^{1/2} & , 0 \leq n_r \leq 0.5 \\ \infty & , 0.5 \leq n_r \leq 1 \end{cases} \tag{29}$$

where $n_r < 0.5$ and $n_r > 0.5$ are used in (26) and (27) respectively. Equations (26) and (27) are plotted in Figures 2 and 3 for various indices n, all plots of which are continuous in the entire range $0 \leq n_r \leq 1$. The curve that runs through the point (1,1) in Figure 2 is the Newtonian result from (6). It is clear from Figure 3 that the acceleration diverges in the range $0.5 \leq n_r \leq 1$ for the limit $n \to \infty$. In the range $0 \leq n_r \leq 0.5$ the acceleration is given by the upper equation in (29) — this result is identical with the static black-hole acceleration [3, p. 43].

4 Summary and Comments

The nature of the vacuum state provides a force constraint $(n_r \leq 1)$ on any theory of gravity, whether it's the Newtonian theory or the general theory of relativity [2]. This effect manifests itself rather markedly in the equation for the Kerr-Newman black-hole area A for a charged spinning mass [4]:

$$A = \frac{4\pi G}{c^4} \times \left[2m^2G - Q^2 + 2(m^4G^2 - c^2J^2 - m^2Q^2G)^{1/2}\right] \tag{30}$$

where Q and J are the charge and angular momentum of the mass m. Using the relation in (3) and $G = e_*^2/m_*^2$ [1], it is straightforward to transform (30) into the following equation

$$\begin{aligned} \frac{A}{4\pi r_*^2} = \; & 2\left(\frac{m}{m_*}\right)^2 - \left(\frac{Q}{e_*}\right)^2 + \\ & +2\left[\left(\frac{m}{m_*}\right)^4 - \left(\frac{J}{r_* m_* c}\right)^2 - \left(\frac{m}{m_*}\right)^2\left(\frac{Q}{e_*}\right)^2\right]^{1/2} \end{aligned} \tag{31}$$

where all of the parameters (e_*, m_*, r_*, except c of course) in the denominators of the terms are PV parameters; and all of the terms are properly normalized to the PV state, the area A by the area $4\pi r_*^2$, the angular momentum J by the angular momentum $r_* m_* c$, and so forth.

The "dogleg" in Figure (4) at the Schwarzschild radius r_s $(n_r = 0.5)$ and the pseudo-singularity in the black-hole metric at r_s are features of the Einstein differential geometry approach to relativistic gravity — how realistic these features are remains to be seen. At this point in time, though, astrophysical measurements have not yet reached the $n_r = 0.5$ point (see below) where the dogleg and the black-hole results can be experimentally checked, but that point appears to be rapidly approaching. Whatever future measurements might show, however, the present calculations indicate that the point-mass-PV interaction that leads to n_r may point to the physical mechanism that underlies gravity phenomenology.

The evidence for black holes with all m/r ratios appears to be growing [3, Ch. 6]; so it is important to see if the present calculations can explain the experimental black-hole picture that is prevalent in today's astrophysics. The salient feature of a black hole is the event horizon [3, pp. 2, 152], that pseudo-surface at $r = r_s$ at which strange things are supposed to happen. A white dwarf of mass 9×10^{32}gm and radius 3×10^8cm exerts a curvature force on the PV equal to 2.7×10^{45}dyne, while a neutron star of mass 3×10^{33}gm and radius 1×10^6cm exerts a force of 2.7×10^{48}dyne [2]. Dividing these forces

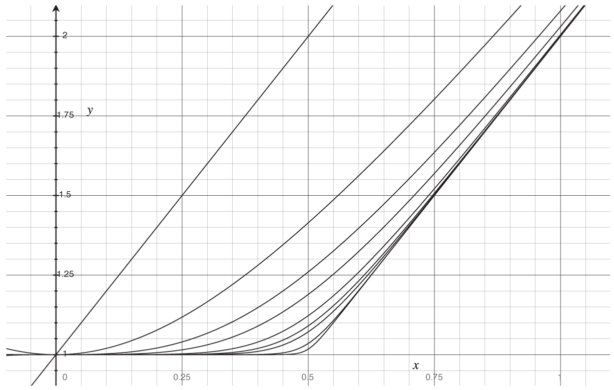

Fig. 4: The graph plots R_n/r as a function of n_r for the indices $n = 1, 2, 3, 4, 6, 8, 10, 20, 40$. The straight line is the $n = 1$ curve ($0 \leq n_r \leq 1$).

by the 1.21×10^{49}dyne force in the denominator of (4) leads to the n-ratios $n_r = 0.0002$ and $n_r = 0.2$ at the surface of the white dwarf and neutron star respectively. The surfaces of these two objects are real physical surfaces — thus they cannot be black holes.

On the other hand, SgrA* [3, p. 156] is thought to be a supermassive black-hole with a mass of about 4.2×10^6 solar masses and a radius confined to $r < 22 \times 10^{11}$ [cm], leading to the SgrA* n-ratio $n_r > 0.28$. For an n-ratio of 0.28, however, the plots in Figures 2–4 show that the behavior of spacetime and the PV is smooth. To reach the $n_r = 0.5$ value and the dogleg, the SgrA* radius would have to be about 12×10^{11} [cm], a result not significantly out of line with the measurements.

Finally, it should be noted that the black-hole formalism is the result of substituting $R_n = r$ in the metric (A1) of the Appendix. Unfortunately, since $R_n/r > 1$ signifies a response of the vacuum to the perturbation n_r at the coordinate radius r, the effect of this substitution is to eliminate that response. This is tantamount to setting $n_r = 0$ in the second-to-last expression of (A3).

Appendix: Crothers Vacuum Metrics

The general solution to the Einstein vacuum field [5] [6] for a point mass m at $r = 0$ consists of the infinite collection ($n = 1, 2, 3, \cdots$) of Schwarzschild-like metrics that are *non-singular* for all $r > 0$:

$$ds^2 = (1 - r_s/R_n)\, c^2 dt^2 - \frac{(r/R_n)^{2n-2}\, dr^2}{1 - r_s/R_n} - \\ -R_n^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{A1}$$

where

$$r_s = 2\,\frac{mG}{c^2} = 2\,\frac{mc^2}{m_* c^2/r_*} = 2rn_r \tag{A2}$$

$$R_n = (r^n + r_s^n)^{1/n} = r(1 + 2^n n_r^n)^{1/n} = r_s \frac{(1 + 2^n n_r^n)^{1/n}}{2n_r} \tag{A3}$$

and where r is the coordinate radius from the point mass to the field point of interest and r_s is the Schwarzschild radius. The ratio R_n/r as a function of n_r is plotted in Figure 4 for various indices n. The n-ratios $0, 0.5$, and 1 correspond to the r values $r \to \infty$, r_s, and $r_s/2$ respectively.

All the metrics in (A1) for $n \geq 2$ reduce to

$$ds^2 = (1 - 2n_r)\, c^2 dt^2 - \frac{dr^2}{1 - 2n_r} - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{A4}$$

for $n_r \ll 1$.

It is clear from the expressions in (A3) that the requirement of asymptotic flatness [3, p.55] is fulfilled for all finite n. On the other hand, the proper radius $\mathcal{R}_n$ from the point mass at $r = 0$ to the coordinate radius r is not entirely calculable:

$$\mathcal{R}_n(r) = \int_0^r (-g_{11})^{1/2} dr$$
$$= \int_0^{r_s/2} (?) dr + \int_{r_s/2}^r (-g_{11})^{1/2} dr \tag{A5}$$

due to the failure of the general theory in the region $0 < r < r_s/2$ [2].

The metric coefficients of interest in the text for $d\theta = d\phi = 0$ are

$$g_{00} = (1 - r_s/R_n) \tag{A6}$$

$$g_{11} = -\frac{(r/R_n)^{2n-2}}{1 - r_s/R_n} = \frac{1}{g^{11}}\,. \tag{A7}$$

From (A3)

$$\frac{\partial R_n}{\partial r} = \frac{1}{(1 + 2^n n_r^n)^{(1-1/n)}} \tag{A8}$$

and from (A8)

$$\frac{\partial g_{00}}{\partial r} = \frac{r_s}{R_n^2}\frac{\partial R_n}{\partial r}\,. \tag{A9}$$

Submitted on May 3, 2012 / Accepted on May 11, 2012

References

1. Daywitt W.C. The Planck vacuum. *Progress in Physics*, 2009, v. 1, 20–26.
2. Daywitt W.C. Limits to the validity of the Einstein field equations and general relativity from the viewpoint of the negative-energy Planck vacuum state. *Progress in Physics*, 2009, v. 3, 27–29.
3. Raine D., Thomas E. *Black Holes: An Introduction*. Second Edition, Imperial College Press, London, 2010. The reader should note that m is used in the present paper to denote physical mass — it is not the geometric mass $m\,(\equiv MG/c^2)$ as used in this reference.
4. http://www.phy.olemiss.edu/~luca/Topics/bh/laws.html
5. Crothers S.J. On the general solution to Einstein's vacuum field and its implications for relativistic degeneracy. *Progress in Physics*, 2005, v. 1, 68–73.
6. Daywitt W.C. The Planck vacuum and the Schwarzschild metrics. *Progress in Physics*, 2009, v. 3, 30–31.