The Dirac Plane Wave[1]

The Dirac equation has been interpreted as a single-particle equation with positive and negative energies ever since its inception in 1930 [1]. Replacing the Planck constant (a secondary constant) in that equation with its corresponding primary constants makes the equation understandable and leads to a different conclusion. The calculations to follow show that the Dirac equation represents a massive point charge interacting with a negative-energy vacuum state—that vacuum state turns out to be the degenerate Planck vacuum (PV) [2]. The plane-wave solution to the Dirac equation ties the Dirac electron to the Compton and de Broglie wavelengths and the Minkowski (flat) spacetime. A brief look at the propagator theory shows that the quantum vacuum and the PV are related.

1 Introduction

How can the point electron (a massive point charge) have Compton and a de Broglie wavelengths? The common-sense answer to the question is that the point electron as an *independent* entity cannot possess a wavelength. Wave-particle duality aside, our scientific intuition rebels against such a concept. To have a wave-like nature, the point electron must be coupled to some continuum that can support such a property. The following sections highlight that coupling and connect the resulting Dirac equation to the PV and, using the plane-wave solution to the equation, connect the corresponding Dirac electron to its Compton and de Broglie wavelengths.

The Conclusions and Comments Section includes a brief mention of the propagator (Green-function) solution to the Dirac equation, showing that the propagator is tied to the PV and that the quantum vacuum and the PV are related.

2 Dirac Equation

In its rest frame, the massless bare charge $(-e_*)$ perturbs the PV state with a radial coupling force equal to e_*^2/r^2 [2]. Consequently, when the charge moves at a uniform velocity in the laboratory frame, it (and the PV) generate the well-known relativistic electric and magnetic fields and the Lorentz transformation connecting the two (static and moving) frames. One of the charges $(-e_*)$ in the product e_*^2 belongs to the free charge and the other $(-e_*)$ to each of the individual Planck particles making up the PV.

For the massive point charge $(-e_*, m)$ in its rest frame, the perturbing force is two-fold however:

$$\frac{e_*^2}{r^2} - \frac{mc^2}{r} = \frac{e_*^2}{r^2}\left(1 - \frac{r}{r_c}\right) \tag{1}$$

[1]To be published in *Galilean Electrodynamics.*

where m is the electron mass. This force vanishes at the electron's Compton radius and leads to [2]

$$c\hbar = e_*^2 = r_c mc^2 \tag{2}$$

where $r_c \,(= e_*^2/mc^2)$ is the Compton radius. Thus the PV is unstressed on a sphere of radius $r = r_c$ surrounding the point electron. For $r > r_c$ the force compresses the vacuum and for $r < r_c$ the vacuum is forced to expand.

That the second term in the perturbation force is an electron-PV interaction force can be seen by rewriting it in the following manner

$$\frac{mc^2}{r} = \frac{mc^2}{r}\frac{G}{e_*^2/m_*^2} = \frac{mc^2}{r}\frac{m_*^2 G}{r_* m_* c^2} = \frac{m m_* G}{r_* r} \tag{3}$$

where $G = e_*^2/m_*^2$ (Newton's gravitational constant) and (2) have been used to arrive at the final ratio which represents a gravitational attraction between the free point electron and the Planck particles in the PV at a radius r from the electron.

Strictly speaking, the curvature force (mc^2/r) in (1) warps spacetime and the PV, but the relative magnitude of that warp is so vanishingly small [2]

$$n(r) = \frac{mc^2/r}{m_* c^2/r_*} \sim \frac{10^{-55}}{r} \tag{4}$$

for any reasonably sized radius ($r \gg r_* \sim 10^{-33}$[cm]), that the resulting spacetime is for all practical purposes flat.

The Dirac equation is [3, p.74]

$$i\hbar\frac{\partial\psi}{\partial t} = c\vec{\alpha}\cdot\frac{\hbar\nabla\psi}{i} + \beta' mc^2\psi \tag{5}$$

where the details of the matrix operators $\vec{\alpha}$ and β' are given in the reference, but are unimportant here. Using (2), (5) can be expressed as

$$ic\hbar\left(\frac{\partial\psi}{\partial ct} + \vec{\alpha}\cdot\nabla\psi\right) = \tag{6}$$

$$ie_*^2\left(\frac{\partial\psi}{\partial ct} + \vec{\alpha}\cdot\nabla\psi\right) = \beta' mc^2\psi \tag{7}$$

and, dividing through by mc^2

$$i\left(\frac{\partial}{\partial ct/r_c} + \vec{\alpha}\cdot(r_c\nabla)\right)\psi = \beta'\psi \tag{8}$$

where the two differential operators on the left in (8) are now scaled or normalized by the Compton radius r_c. The corresponding Compton wavelength is $\lambda_c = 2\pi r_c$.

3 Plane-Wave Solution

The ψ in (5)—(8) is a four-component spinor. The positive-energy-spin-up plane-wave solution for ψ (with the electron traveling at a uniform velocity v along the positive z-axis) can be expressed as [3, p.89]

$$\psi_1^{(+)} = u(p,1) \exp\left[-i\left(\frac{Et - pz}{\hbar}\right)\right] \tag{9}$$

$$= u(p,1) \exp\left[-i\left(\frac{m\gamma c \cdot ct - m\gamma v \cdot z}{\hbar}\right)\right] \tag{10}$$

$$= u(p,1) \exp\left[-i\left(\frac{ct}{r_c/\gamma} - \frac{z}{r_c/\beta\gamma}\right)\right] \tag{11}$$

where

$$u(p,1) = N_p \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ B_p \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \tag{12}$$

$$N_p \equiv \left(\frac{E + mc^2}{2mc^2}\right)^{1/2} = \left(\frac{\gamma + 1}{2}\right)^{1/2} \tag{13}$$

$$B_p \equiv \frac{cp}{E + mc^2} = \frac{\beta\gamma}{\gamma + 1} \tag{14}$$

$E = (m^2c^4 + c^2p^2)^{1/2} = m\gamma c^2$, $p = m\gamma v$, $\beta = v/c$, and $\gamma = 1/\sqrt{1-\beta^2}$. Equation (2) is used to obtain the secondary results. The four-momentum $p^\mu = (m\gamma c, 0, 0, m\gamma v)$ in (10) leads to

$$\frac{(p^\mu p_\mu)^{1/2}}{\hbar} = \frac{[(m\gamma c, 0, 0, m\gamma v) \cdot (m\gamma c, 0, 0, -m\gamma v)]^{1/2}}{\hbar} = \frac{mc}{\hbar} = \frac{1}{r_c}\,. \tag{15}$$

4 Conclusions and Comments

The appearance of the coupling constants (e_*^2 and mc^2) from (1) in the Dirac equation (7) implies that the Dirac equation represents a response of the PV to the motion of the point electron. This effect can also be concluded from (8) which shows that the spacetime coordinates

$$x^\mu = (x^0 = ct, x^1, x^2, x^3) \qquad \longrightarrow \qquad \left(\frac{ct}{r_c}, \frac{x^1}{r_c}, \frac{x^2}{r_c}, \frac{x^3}{r_c}\right) \tag{16}$$

are scaled by the Compton radius, where r_c represents the ratio (e_*^2/mc^2) of the two coupling constants and the radius from the point electron where the PV remains unstressed.

The plane-wave solution in (10) shows that the two momenta $m\gamma c$ and $m\gamma v$ propagate along the positive ct- and z-axes respectively. Furthermore, the r_c

in the denominators of the exponential in (11) is a result of the scaling in (16). Those denominators

$$r_L \equiv \frac{r_c}{\gamma} \qquad \text{and} \qquad r_d \equiv \frac{r_c}{\beta\gamma} \tag{17}$$

are respectively the length-contracted r_c associated with the ct-axis and the de Broglie radius r_d associated with the z-axis. The corresponding de Broglie wavelength is $\lambda_d = 2\pi r_d$.

The Minkowski-spacetime configuration established by the preceding calculations is clear: the electron ray path is normalized by the Compton radius r_c [see (15)]; the projection of this radius onto the ct-axis is r_L; and the projection onto the z-axis is the de Broglie radius r_d. Synge [4] referred to the above scaling as the 'primitive quantization' of spacetime—the calculations here provide a physical basis for that viewpoint.

Of the two approaches for calculating transition rates and cross-sections in scattering problems [3, p.143], the quantum field approach depends heavily on the idea of a vacuum state—and yet the propagator approach seems to ignore the vacuum altogether. The brief calculations to follow resolve that apparent asymmetry in viewpoints.

The propagator equation can be expressed as [3, p.170]

$$\left(ic\hbar\gamma_\mu \frac{\partial}{\partial x'_\mu} - mc^2 \right) S_F(x';x) \tag{18}$$

$$= \left(ie_*^2\gamma_\mu \frac{\partial}{\partial x'_\mu} - mc^2 \right) S_F(x';x) = \delta^4(x'-x) \tag{19}$$

where $S_F(x';x)$ is the Feynman propagator. There is nothing in (18) to suggest that this propagator is connected to a vacuum state. Using (2) in (18) leads to equation (19) which is more revealing however. When it is recognized that the two constants e_*^2 and mc^2 in (19) are the two coupling constants from the interaction force (1), then it is clear that the propagator must be associated with the PV state.

The quantum-vacuum (QV) propagator is defined by the equation [5, p.373]

$$iS_F(x',x) = \langle 0|T\psi(x')\overline{\psi}(x)|0\rangle \tag{20}$$

where $\langle 0|$ and $|0\rangle$ are the bra and ket vectors denoting the QV state, the ψs represent quantum fields, and T is the time-ordering operator. As opposed to (18), the propagator in (20) is clearly associated with a vacuum state due to the presence of the bra and ket vectors. Furthermore, using a plane-wave expansion for the ψ operators [5, p.349], the QV propagator can be reduced to the propagator in (19). Therefore (19) and (20) are equivalent and thus the QV and the PV are related!

It is noted in closing that the negative energy levels of the QV state have no lower bound [5, p.354], while the PV energy-level descent is bounded below by the Planck energy ($-e_*^2/r_* = -m_*c^2$).

References

[1] P.A.M. Dirac, "A Theory of Electrons and Protons". Proc. Roy. Soc. Lond., A **126**, p.360 (1930).

[2] W.C. Daywitt, "The Planck Vacuum". Galilean Electrodynamics, v. 21, no. 4, p.72 (2010).

[3] D.M. Gingrich, **Practical Quantum Electrodynamics** (The Taylor & Francis Group, Boca Raton, London, New York, 2006).

[4] J.L.Synge, **Geometrical Mechanics and de Broglie Waves** (Cambridge University Press, 1954). See p.106.

[5] P.M. Milonni, **The Quantum Vacuum—an Introduction to Quantum Electrodynamics** (Academic Press, New York, 1994).