The de Broglie Wavelength and the Dirac Equation

Following the analogy of the photon wave-particle duality, L. de Broglie in 1924 proposed that electrons might have wave-like properties if the wavelength and momentum of the electron are related by the equation [1, p.81]

$$\lambda = \frac{h}{p}$$

(1)

where $h$ is Planck’s constant and $p$ is the electron momentum. With the development of the Planck vacuum (PV) model [2], that proposal can be explained in depth. Results provide a complete Minkowski-spacetime picture of the de Broglie relation in (1) and its connection to the Compton relation and the Dirac equation.

1 de Broglie Relationship

In its rest frame the Dirac electron (a massive point charge) exerts the two-fold perturbation force [2]

$$\frac{e^2}{r^2} - \frac{mc^2}{r} = \frac{e^2}{r^2} \left(1 - \frac{r}{r_c}\right)$$

(2)

on each of the Planck particles within the degenerate PV, where the first and second terms in the force difference represent polarization and curvature forces respectively. The electron is characterized by $(-e_*, m)$ where $-e_*$ and $m$ are the electron’s bare charge and mass, the bare charge and the observed electronic charge $e$ being related through the fine structure constant $\alpha = e^2/e^2_*$.

The perturbation force vanishes

$$\frac{e^2}{r_c^2} - \frac{mc^2}{r_c} = 0$$

(3)

at the electron Compton radius

$$r_c = \frac{e^2_*}{mc^2}$$

(4)

an important constant in the electron’s dynamical behavior. Using $ch = e^2_*$ expresses the Compton relation (4) in the form

$$mc = \frac{h}{r_c} = \frac{h}{\lambda_c}$$

(5)

where $\lambda_c = 2\pi r_c$ is the Compton wavelength.

The vanishing of the force difference (3) is a Lorentz-invariant constant which, after transformation from its rest frame to a uniformly moving laboratory frame, leads to the two equations [2]:

$$\gamma \left(\frac{e^2}{r_c^2} - \frac{mc^2}{r_c}\right) = 0$$

and

$$\beta\gamma \left(\frac{e^2}{r_c^2} - \frac{mc^2}{r_c}\right) = 0$$

(6)
where if \( v \) is the electron velocity, then \( \beta = v/c \) and \( \gamma = \sqrt{1/(1-\beta^2)} \). If \( 0 < \beta < 1 \), then the equations in (6) yield the two de Broglie relations:

\[
m\gamma c = \frac{\hbar}{r_L} \quad \text{where} \quad r_L \equiv \frac{r_c}{\gamma} \tag{7}
\]
is the length-contracted \( r_c \) in the \( ct \)-direction in spacetime; and

\[
p = m\gamma v = \frac{\hbar}{r_d} \quad \text{where} \quad r_d \equiv \frac{r_c}{\beta\gamma} \tag{8}
\]
is the de Broglie radius and \( p \) is the electron’s relativistic momentum. The de Broglie wavelength is

\[
\lambda_d = 2\pi r_d \tag{9}
\]
which with (8) leads back to the original de Broglie 1924 proposal in (1). Thus the de Broglie relations (7) and (8) are the result of the electron perturbation (2) and the Lorentz transformation, both relations involving the constant \( \gamma \). Combining the two equations in (7) and those in (8) shows, of course, that the first equations in both (7) and (8) are just disguised versions of the Compton relation (5).

The momentum \( m\gamma c \) along the \( ct \)-axis and the momentum \( m\gamma v \) along the spatial axes are both proportional to ‘Planck’s constant divided by a wavelength’; so the single de Broglie relation in (1) is formally replaced by the two de Broglie relations in (7) and (8) and represent a complete spacetime picture of the de Broglie relationship.

2 Dirac Equation

Using \( c\hbar = e^2 \) for Planck’s constant, the Lorentz-covariant Dirac equation can be expressed as [3, p.90]

\[
\left( i\hbar \gamma^\mu \frac{\partial}{\partial x_\mu} - mc^2 \right) \Psi = 0 \quad \rightarrow \quad \left( i\gamma^\mu \frac{\partial}{\partial x_\mu} - 1 \right) \Psi = 0 \tag{10}
\]
where \( \Psi \) is the four-component Dirac spinor and where the four coordinates \((x_0 = ct, x_1, x_2, x_3)\) are normalized in the final equation by the Compton radius. It is noted that \( r_c \) is the only dynamical constant in that equation.

For the plane-wave solution to (10) with the electron traveling at a uniform velocity along the positive \( x_3 = z \) axis, the first of the four spinor components in \( \Psi \) can be expressed as [3, p.89] [2] \( \hbar = r_c mc \) is used in obtaining (11)

\[
\psi_1^{(+)} = N_p \begin{pmatrix} 1 & 0 \\ B_p & 1 \end{pmatrix} \exp \left[ -i \left( \frac{Et - pz}{\hbar} \right) \right]
\]

2
\[ N_p \begin{pmatrix} 1 \\ 0 \\ B_p \end{pmatrix} \exp \left[ -i \left( \frac{ct}{r_c/\gamma} - \frac{z}{r_c/\beta\gamma} \right) \right] \]  \hspace{1cm} (11)

where

\[ N_p = \left( \frac{E + mc^2}{2mc^2} \right)^{1/2} \quad \text{and} \quad B_p = \frac{cp}{E + mc^2} \]  \hspace{1cm} (12)

and \( E = (m^2c^4 + c^2p^2)^{1/2} = m\gamma c^2 \), and where the coordinates \( ct \) and \( z \) are normalized by the radii \( r_L \) and \( r_d \) from (7) and (8).

Summarizing then, the radii \( r_c, r_c/\gamma, \) and \( r_c/\beta\gamma \) in (10) and (11) tie the Dirac equation directly to the Compton relation in (5) and the de Broglie relations in (7) and (8), and imply that the Dirac equation represents a PV response to the perturbation in (2). It is interesting to compare this spacetime quantization \( (r_c, r_L, r_d) \) with the ‘primitive quantization’ of spacetime in terms of de Broglie 3-waves provided by Synge [4].

References


